Sums of toric ideals

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Abstract

Given two toric ideals $I_1, I_2 \subset \Bbbk[\mathbf{x}]$, it is not always true that $I_1 + I_2$ is a toric ideal. Given $I_1, \dots, I_k \subset \Bbbk[\mathbf{x}]$ a familly of toric ideals we give necessary conditions in order to have that $I_1 + \cdots + I_k$ is a toric ideal.

1 Introduction

Let $\Bbbk$ be a field. Let $\Bbbk[\mathbf{t}^{\pm}] := \Bbbk[\mathbf{t}_1^{\pm}, \dots, \mathbf{t}_m^{\pm}]$ the Laurent polynomial ring on m variables over a field $\Bbbk$. Let $\alpha = (\alpha^{(1)}, \dots, \alpha^{(m)}) \in \mathbb{Z}^m$, $\mathbf{t}^{\alpha} = t_1^{\alpha^{(1)}} \dots t_m^{\alpha^{(m)}}$ is called a monomial of $\Bbbk[\mathbf{t}]$. Let $Y_1 = \mathbf{y}^{\alpha_1}, \dots, Y_n = \mathbf{y}^{\alpha_n}$ be monomials in $\Bbbk[\mathbf{t}]$, we denote by $S := \Bbbk[Y_1, \dots Y_m]$ the subalgebra of the polynomial ring over the field k generated by $Y_1, \dots, Y_m$. The set of monomials $\mathcal{M}$ contained in S form a semigroup under multiplication, and the function $\deg : \mathcal{M} \to \mathbb{N}^m$ that assigns each monomial its exponent vector maps $\mathcal{M}$ isomorphically onto a subsemigroup H of $\mathbb{N}^m$. Up to isomorphism, S is the semigroup algebra $\Bbbk[H]$. We will always assume that $Y_1, \dots, Y_n$ are irreducible elements of $\mathcal{M}$, or, in other words, that they form a minimal system of algebra generators of S.

Now, let $\Bbbk[\mathbf{x}] := \Bbbk[x_1, \dots, x_n]$ the polynomial ring on n-variables and $\varphi : \Bbbk[\mathbf{x}] \to S$ the algebra morphism defined by $\varphi(x_i) = Y_i$. As S is a Laurent polynomial ring, S is an integer domain and $J = \text{Ker } \varphi$ is prime. We will say that J is a **toric ideal** of $\Bbbk[\mathbf{x}]$ and φ is a parametrization of J. The next two propositions are well known:

1.1 Proposition. *Let $A = [\alpha_1, \dots, \alpha_n] \in M_{m,n}(\mathbb{Z})$, $\pi : \mathbb{N}^n \to \mathbb{Z}^m$ the semigroup morphims defined by $\pi(\mathbf{e}_i) = \alpha_i$ and $\varphi : \Bbbk[\mathbf{x}] \to \Bbbk[t^{\alpha_1}, \dots, t^{\alpha_n}]$ the algebra morphism defined by $\varphi(x_i) = t^{\alpha_i}$. Then* $\ker \varphi = I_A := (\mathbf{x}^{u_+} - \mathbf{x}^{u_-} : \pi(u_+) = \pi(u_-))$. *Moreover, for any $u \in \mathbb{N}^n$, $\varphi(z^u) = t^{A \cdot u}$.*

1.2 Proposition. *Let $\varphi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{t}^{\pm 1}]$ be a parametrization of $I := \text{Ker } \varphi$ such that $\varphi(x_i) = t^{\alpha_i}$. Let set $A = [\alpha_1, \dots, \alpha_n] \in M_{m,n}(\mathbb{Z})$, then* $\text{rank } (A) = \dim\, (\Bbbk[\mathbf{x}]/I)$.

1.3 Definition. Let $\varphi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{t}^{\pm 1}]$ a parametrization of $I := \text{Ker } \varphi$. We say that φ has maximal rank if $\dim\, (\Bbbk[\mathbf{x}]/I)$ is equal to cardinality of $\mathbf{t}$.

We say that $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ and $v = (v_1, \dots, v_n) \in \mathbb{N}^n$ have disjoint support if for all $i \in \mathbb{N}$ such that $u_i \neq 0$ then $v_i = 0$ and reciprocally. Furthermore, for any $u \in \mathbb{Z}^n$ there is two unique vectors $u_+, u_- \in \mathbb{N}^n$ with disjoint supports such that $u = u_+ - u_-$. In this manner, by proposition 1.1 we deduce the next proposition:

1.4 Proposition. *Let $\varphi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{t}^{\pm 1}]$ be a parametrization of* Ker φ. *Then, setting*

$$\text{Ker}_{\mathbb{Z}}\ A := \{u \in \mathbb{Z}^n : Au = 0\},$$

$\text{Ker } \varphi = (\mathbf{x}^{u_+} - \mathbf{x}^{u_-} : u_+, u_- \in \mathbb{N}^n$ of disjoint support and $(u_+ - u_-) \in \text{Ker}_{\mathbb{Z}}\ A)$. ■

From the proposition 1.4, in order to describe a toric ideal I, it is enough to solve the linear homogene equation system over $\mathbb{Z}$

$$A \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = 0,$$

where A is a matrix related to the parametrization φ of I. We know from linear algebra that for any $Q \in M_{m,m}(\mathbb{Q})$ non-singular, $\mathrm{Ker}_{\mathbb{Z}}\ A = \mathrm{Ker}_{\mathbb{Z}}\ QA$, thus the next proposition is immediate.

1.5 Proposition. *If $\widetilde{A} \in M_{m,n}(\mathbb{Z})$ is a matrix obtained from A by a left multiplication for a non-singular matrix $Q \in M_{m,m}(\mathbb{Q})$, then $I_{\widetilde{A}} = \mathrm{Ker}\ \varphi$ and the matrix $\widetilde{A}$ define a new parametrization of $\mathrm{Ker}\ \varphi$ par $\widetilde{\varphi}(x^u) = \mathbf{t}^{[\widetilde{A}]\cdot u}$. In particular, if $\widetilde{A} \in M_{m,n}(\mathbb{Z})$ is obtained by a finite sequence of elementary operations over the lines of A, then $I_{\widetilde{A}} = \mathrm{Ker}\ \varphi$ and the matrix B define a new parametrization of $\mathrm{Ker}\ \varphi$. Reciprocally, if $I \subset \Bbbk[x_1, \ldots, x_n]$ is a toric ideal, then $L = \{u \in \mathbb{Z}^n : x^{u_+} - x^{u_-} \in I\}$ is a saturated lattice. If $\{b_1, \ldots, b_m\}$ is a basis of L and $B = [b_1 \ldots b_m] \in M_{n,m}$ then any matrix $A \in M_{m,n}(\mathbb{Z})$ such that $AB = 0$, A is a parametrization of I.*

■

2 Sums of toric ideals

2.1 Définition. We consider two polynomial rings $\Bbbk[\mathbf{z}] := \Bbbk[z_1, \ldots, z_n]$ and $\Bbbk[x][z] \cong \Bbbk[x, z_1, \ldots, z_n]$. Let $f := f(z_1, \ldots, z_n)$ be a polynomial in $\Bbbk[\mathbf{z}]$ and $g := g(z_1, \ldots z_n, x)$ a homogeneous polynomial in $\Bbbk[\mathbf{x}][\mathbf{z}]$. Writting $f = \sum_{i=1}^m a_i \mathbf{z}_i^\beta$ for some $a_i \in \Bbbk$ and $\beta_i \in \mathbb{N}^n$, we define the degree of f as $\deg_{\mathbb{N}}(f) = \max_{i=1}^n(|\beta_i|)$. The **homogenization** of f by x is the polynomial $f^{\mathrm{hom}_x} = x^{\deg_{\mathbb{N}}(f)} f(\frac{z_1}{x}, \ldots, \frac{z_n}{x})$. The **dehomogenization** of g by x is the polynomial $g^{\mathrm{deh}_x} = g(z_1, \ldots z_n, 1) \in \Bbbk[\mathbf{z}]$. In case where there is not confusion about the variable x, we will only write f^{hom} and g^{deh}. In addition, let I be an ideal of $\Bbbk[\mathbf{z}]$ and J an ideal of $\Bbbk[\mathbf{z}, x]$. The homogenization of I by x is the ideal $I^{\mathrm{hom}} = (f^{\mathrm{hom}} : f \in I)$ and the dehomogenization of J by x is the ideal $J^{\mathrm{deh}} = (g^{\mathrm{deh}} : g \in J)$.

2.2 Lemma. [17, Lemma 4.14] *Let $A = [\alpha_1, \ldots, \alpha_n] \in M_{m,n}(\mathbb{Z})$. The ideal I_A is homogeneous if and only if there exists a vector $\omega \in \mathbb{Q}^m$ such that $\alpha_i \cdot \omega = 1$ for any $i = 1, \ldots, n$ for which $\alpha_i \neq 0$.*

2.3 Lemma. *Let $\mathbf{z} = (z_1, \ldots, z_n)$ be a sequence of variables, x be a variable which does not belong to $\mathbf{z}$ and $I \subset \Bbbk[\mathbf{z}, x]$ be a homogeneous toric ideal such that there exist a parametrization of I:*

$$\varphi : \Bbbk[\mathbf{z}, x] \to \Bbbk[\mathbf{t}^{\pm 1}, s]$$

with $\varphi(z_i) = \mathbf{t}^{\alpha_i} s^{\gamma_i}$ where $\alpha_i \in \mathbb{Z}^n$, $\gamma_i \in \mathbb{Z}$, and $\varphi(x) = s^{\gamma_{n+1}}$ where $\gamma_{n+1} \in \mathbb{Z}^*$. Let $\widetilde{\varphi} : \Bbbk[\mathbf{z}] \to \Bbbk[\mathbf{t}^{\pm 1}]$ be the morphism defined by $\widetilde{\varphi}(z_i) = (\varphi(z_i))^{\mathrm{deh}_s}$, then $(\mathrm{Ker}\ \widetilde{\varphi})^{\mathrm{hom}} = I$.

Proof. We set $\widetilde{A} = [\alpha_1, \ldots, \alpha_n] \in M_{m,n}(\mathbb{Z})$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ and

$$A = \left(\begin{array}{c|c} \widetilde{A} & \mathbf{0} \\ \hline \gamma & \gamma_{n+1} \end{array}\right) \in M_{m,n}(\mathbb{Z}) \text{ so that } I := \mathrm{Ker}\ \varphi = I_A.$$

Now, we are going to prove that $(\mathrm{Ker}\ \widetilde{\varphi})^{\mathrm{hom}} = I$. Clearly $I \subset (\mathrm{Ker}\ \widetilde{\varphi})^{\mathrm{hom}}$; so we only need to prove that $(\mathrm{Ker}\ \widetilde{\varphi})^{\mathrm{hom}} \subset I$.

Let $\mathbf{z}^{u_+} - \mathbf{z}^{u_-} \in \mathrm{Ker}\ \widetilde{\varphi}$, where $u_+ = (u_+^{(1)}, \ldots, u_+^{(n)})$ and $u_- = (u_-^{(1)}, \ldots, u_-^{(n)}) \in \mathbb{N}^n$ have disjoint support. We should prove that $(\mathbf{z}^{u_+} - \mathbf{z}^{u_-})^{\mathrm{hom}} \in I$. We recall that if $u = (u^{(1)}, \ldots, u^{(n)}) \in \mathbb{Z}^n$, $|u| = \sum_{i=1}^n u^{(i)}$. Without lost of generality we can assume $|u_+| \leq |u_-|$. Setting $u_+^{(n+1)} \in \mathbb{N}$ such that $|u_+| + u_+^{(n+1)} = |u_-|$, we have that

$$(\mathbf{z}^{u_+} - \mathbf{z}^{u_-})^{\mathrm{hom}} = \mathbf{z}^{u_+} x^{u_+^{(n+1)}} - \mathbf{z}^{u_-}.$$

Let set $\widetilde{u}_+ = (u_+, u_+^{(n+1)}) \in \mathbb{N}^{n+1}$, and $\widetilde{u}_- = (u_-, 0) \in \mathbb{N}^{n+1}$. Since I is homogeneous, thanks to lemma 2.2 there exists $\omega \in \mathbb{Q}^m$ and $\omega_{m+1} \in \mathbb{Q}$ such that $(\omega, \omega_{m+1}) \cdot \alpha_i' = 1$ for any $i = 1, \ldots, n+1$, this is equivalent to

$$(\omega, \omega_{m+1}) \cdot A = (1, \ldots, 1).$$

Thus,

$$\begin{aligned} |u_+| + u_+^{(n+1)} &= (1, \cdots 1) \cdot \widetilde{u}_+ \\ &= ((\omega, \omega_{m+1}) \cdot A) \cdot (u_+, u_+^{(n+1)}) \\ &= \omega \cdot (\widetilde{A} \cdot u_+) + \omega_{m+1}((\gamma, \gamma_{n+1}) \cdot \widetilde{u}_+) \\ &= \omega \cdot (\widetilde{A} \cdot u_+) + \omega_{m+1} \textstyle\sum_{i=1}^{n+1} \gamma_i u_+^{(i)}, \end{aligned}$$

and

$$\begin{aligned} |u_-| &= (1, \cdots 1) \cdot \widetilde{u}_- \\ &= ((\omega, \omega_{m+1}) \cdot A) \cdot (u_-, 0) \\ &= \omega \cdot (\widetilde{A} \cdot u_-) + \omega_{m+1} \textstyle\sum_{i=1}^{n} \gamma_i u_-^{(i)}. \end{aligned}$$

Besides $0 = \widetilde{\varphi}(\mathbf{z}^{\mathbf{u}_+} - \mathbf{z}^{\mathbf{u}_-}) = \mathbf{t}^{\widetilde{\mathbf{A}} \cdot \mathbf{u}_+} - \mathbf{t}^{\widetilde{\mathbf{A}} \cdot \mathbf{u}_-}$, so $\widetilde{A} \cdot u_+ = \widetilde{A} \cdot u_-$. Furthermore, $|u_+| + u_{n+1} = |u_-|$, and then $\omega_{m+1} \sum_{i=1}^{n+1} \gamma_i u_+^{(i)} = \omega_{m+1} \sum_{i=1}^{n} \gamma_i u_-^{(i)}$. In addition, $\omega_{m+1} \neq 0$, because

$$1 = (\omega, \omega_{m+1}) \cdot \alpha_{m+1}' = \omega_{m+1} \gamma_{m+1}.$$

Thus,

$$\sum_{i=1}^{n+1} \gamma_i u_+^{(i)} = \sum_{i=1}^{n} \gamma_i u_-^{(i)}. \tag{1}$$

On the other side

$$\begin{aligned} \varphi(\mathbf{z}^{\mathbf{u}_+} \mathbf{x}^{\mathbf{u}_+^{(\mathbf{n+1})}} - \mathbf{z}^{\mathbf{u}_-}) &= \varphi(z^{u_+} x^{u_+^{(n+1)}}) - \varphi(\mathbf{z}^{u_-}) \\ &= (\mathbf{t}, s)^{A \cdot u_+'} - (\mathbf{t}, s)^{A \cdot u_-'} \\ &= \mathbf{t}^{\widetilde{A} \cdot u_+} s^{\gamma \cdot u_+} s^{\gamma_{n+1} u_+^{(n+1)}} - \mathbf{t}^{\widetilde{A} \cdot u_-} s^{\gamma \cdot u_-} \\ &= \mathbf{t}^{\widetilde{A} \cdot u_+} s^{\sum_{i=1}^{n+1} \gamma_i u_+^{(i)}} - \mathbf{t}^{\widetilde{A} \cdot u_-} s^{\sum_{i=1}^{n} \gamma_i u_-^{(i)}} \\ &= \mathbf{t}^{\widetilde{A} \cdot u_+} s^{\sum_{i=1}^{n+1} \gamma_i u_+^{(i)}} - \mathbf{t}^{\widetilde{A} \cdot u_+} s^{\sum_{i=1}^{n} \gamma_i u_-^{(i)}} \\ &= \mathbf{t}^{\widetilde{A} \cdot u_+} (s^{\sum_{i=1}^{n+1} \gamma_i u_+^{(i)}} - s^{\sum_{i=1}^{n} \gamma_i u_-^{(i)}}) = 0. \end{aligned}$$

The last inequality is due to the equation 1. By consequence

$$(\mathbf{z}^{u_+} - \mathbf{z}^{u_-})^{\text{hom}} = \mathbf{z}^{u_+} x^{u_+^{(n+1)}} - \mathbf{z}^{u_-} \in I$$

and $(\text{Ker } \widetilde{\varphi})^{\text{hom}} \subset I$. As we have already seen $I \subset (\text{Ker } \widetilde{\varphi})^{\text{hom}}$, we conclude $(\text{Ker } \widetilde{\varphi})^{\text{hom}} = I$.

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2.4 Lemma. *Let $A \in M_{m,n}(\mathbb{Z})$ where $m \leq n$ and* rang $(A) = m$. *Then, for any $i \in \{1, \ldots, n\}$ such that the i-th column of A is not nul, there always exists a submatrix A' of m columns of A where the i-th column of A is a column of A' and A' is non-singular over $\mathbb{Q}$.*

Proof. Since rank $(A) = m$, there exists a matrix $A_1 \in M_{m,m}(\mathbb{Z})$ with m columns of A such that A_1 is inversible. Let α_i be the i-th column of A with $\alpha_i \neq 0$ and $\alpha_1', \ldots, \alpha_m' \in \mathbb{Z}^m$ the m-columns of A_1. Then $\{\alpha_1', \ldots, \alpha_m'\}$ is a basis of $\mathbb{Q}^m$. Thus, there exists $a_1, \ldots, a_m \in \mathbb{Q}$ not all nul such that $\alpha_i = \sum_{j=1}^m a_j \alpha_j'$. We can assume $\alpha_1 \neq 0$, then $\alpha_1' = \sum_{j=2}^m \frac{a_j}{\alpha_1} \alpha_j' - \frac{1}{a_1} a_i$. So $\{\alpha_i, \alpha_2', \ldots, \alpha_m'\}$ is a basis of $\mathbb{Q}^m$ and the matrix $A' = [\alpha_i, \alpha_2', \ldots, \alpha_m']$ is non-singular over $\mathbb{Q}$.

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2.5 Lemma. *Let $\varphi : \Bbbk[\mathbf{x}] \to \Bbbk[t_1, \ldots, t_{m'}, t_1^{-1}, \ldots, t_{m'}^{-1}]$ be a parametrization of the ideal $I :=$* Ker φ *and $A^* \in M_{m',n}(\mathbb{Z})$ the matrix related to this parametrization such that*

$$\text{rang } (A^*) = m \leq m'.$$

For any $i \in \{1, \ldots, n\}$ such that $\varphi(x_i) \neq 1$, there exists a parametrization of I

$$\widetilde{\varphi} : \Bbbk[\mathbf{x}] \to \Bbbk[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$$

of maximal rank such that $\widetilde{\varphi}(x_i) = t_j^q$, for some $q \in \mathbb{N}$ and $j \in \{1, \ldots m\}$.

Proof. By hypothesis $\varphi(x_i) = t_1^{\alpha_{i,1}} \cdots t_{m'}^{\alpha_{i,m'}} \neq 1$. Let $j' \in \{1, \ldots, m'\}$ be the smallest integer such that $\alpha_{i,j'} \neq 0$. Let denote by β_r the r-th line vector of the matrix A^*. As $\operatorname{rang}(A^*) = m$, there exist $\beta_{i_1}, \ldots, \beta_{i_m} \in \{\beta_1, \ldots, \beta_{m'}\}$ such that

$$\operatorname{Span}_{\mathbb{Q}}(\beta_{i_1}, \ldots, \beta_{i_m}) \cong \mathbb{Q}^m.$$

Thus there exist $b_1, \ldots, b_m \in \mathbb{Q}$ not all null such that $\beta_{j'} = \sum_{k=1}^m b_k \beta_{i_k}$. We can assume $b_1 \neq 0$, then

$$\beta_{i_1} = \sum_{k=2}^m \frac{b_k}{b_1} \beta_{i_k} - \frac{1}{b_1} \beta_{j'}.$$

So $\beta_{j'}, \beta_{i2}, \ldots, \beta_{im}$ is a basis of $\operatorname{Span}_{\mathbb{Q}}(\beta_{i_1}, \ldots, \beta_{i_m}) = \mathbb{Q}^m$ and there exists a non-singular matrix $B \in M_{m',m'}(Q)$ such that

$$B \cdot A^* = \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix} \in M_{m',n}(\mathbb{Q}), \text{ avec } A = (\beta_{j'}, \beta_{i_{k_2}}, \ldots, \beta_{i_{k_m}})^T \text{ et } \mathbf{0} \in M_{m'-m,n}(\mathbb{Q}).$$

By the lemma 2.4 there exists a non-singular matrix $A' \in M_{m,m}(\mathbb{Z})$ of m columns of A containning the i-th column of A. Moreover, we can assume that $A = [A'|A'']$ where $A'' \in M_{m,n-m}$. Since A' is invertible, there exists $B \in M_{m,m}(\mathbb{Q})$ such that $BA' = I$. By this way $BA = [BA'|BA''] = [I|BA'']$. Let q be the smaller natural integer such that

$$q(BA) = (qB)A = [qI|q(BA'')] \in M_{m,n}(\mathbb{Z}).$$

From the proposition 1.5, $(qB)A$ defines a parametrization of I, $\widetilde{\varphi} : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{t}^{\pm 1}]$ of maximal rank (rank $(A') = m$) such that there exists $j \in \{1, \ldots m\}$ and $\widetilde{\varphi}(x_i) = t_j^q$. ∎

2.6 Theorem. *Let $\mathbf{z_1} = \{z_{1,1}, \ldots, z_{1,n_1}\}$ and $\mathbf{z_2} = \{z_{2,1}, \ldots, z_{2,n_2}\}$ be two disjoint variable sets, x be a new variable and $I_1 \subset \Bbbk[\mathbf{z_1}, x]$, $I_2 \subset \Bbbk[\mathbf{z_2}, x]$ two homogeneous toric ideals such that x appears in at least one generator of a minimal generator system of I_1 and I_2. Then:*

1. *$I_1 + I_2$ is a homogeneous toric ideal of $\Bbbk[\mathbf{z_1}, \mathbf{z_2}, x]$.*
2. $\dim\ (\Bbbk[\mathbf{z_1}, \mathbf{z_2}, x]/(I_1 + I_2)) = \dim\ (\Bbbk[\mathbf{z_1}, x]/I_1) + \dim\ (\Bbbk[\mathbf{z_2}, x]/I_2) - 1$.

Proof. Without lost of generality we can suppose I_1 and I_2 are non degenerated, i.e., for any $i = 1, 2$, any variable of $\mathbf{z_i} \cup \{x\}$ appears in a minimal system of generators of I_i.

Before to prove this theorem, we make the following remarks:
We set $\mathbf{z} = \mathbf{z_1} \cup \mathbf{z_2}$. From the lemma 2.5 there exist

$$\overline{\varphi}_1 : \Bbbk[\mathbf{z_1}, x] \to \Bbbk[\mathbf{t}^{\pm 1}, s^{\pm 1}] \text{ et } \overline{\varphi_2} : \Bbbk[\mathbf{z_1}, x] \to \Bbbk[\mathbf{w}^{\pm 1}, s^{\pm 1}]$$

parametrizations of maximal rank of I_1 and I_2 respectively, where $\overline{\varphi}_i(x) = s^{\gamma_{i,n_i+1}}$ and $\gamma_{i,n_i+1} \in \mathbb{Z}^*$ for any $i = 1, 2$. We set m_1, m_2 the cardinals of $\mathbf{t}$ and $\mathbf{u}$, repectively. We set also $\gamma = \operatorname{lcm}(\gamma_{1,n_1+1}, \gamma_{2,n_2+1})$. We denote by $\overline{A}_1 \in M_{m_1+1,n_1+1}(\mathbb{Z})$ et $\overline{A}_2 \in M_{m_2+1,n_2+1}(\mathbb{Z})$ the matrices which represent the parametrizations $\overline{\varphi_1}$ and $\overline{\varphi_2}$, respectively. These matrices can be written as below:

$$\overline{A}_1 = \left(\begin{array}{c|c} \overline{A}'_1 & \mathbf{0} \\ \hline \alpha'_1 & \gamma_{1,n_1+1} \end{array} \right) \text{ et} \overline{A}_2 = \left(\begin{array}{c|c} \overline{A}'_2 & \mathbf{0} \\ \hline \alpha'_2 & \gamma_{2,n_2+1} \end{array} \right),$$

where $\overline{A}'_1 \in M_{m_1,n_1}(\mathbb{Z})$, $\overline{A}_2 \in M_{m_2,n_2}(\mathbb{Z})$, $\alpha_1 = (\gamma_{1,1}, \ldots, \gamma_{1,n_1}) \in \mathbb{Z}^{n_1}$ and $\alpha_2 = (\gamma_{2,1}, \ldots, \gamma_{2,n_2}) \in \mathbb{Z}^{n_2}$. Multiplying the last line of each matrix $\overline{A}_1$ and $\overline{A}_2$ respectively, by $\frac{\gamma}{\gamma_{1,n_1+1}}$ and $\frac{\gamma}{\gamma_{2,n_2+1}}$, where

$\gamma = lcm(\gamma_{1,n_1+1}, \gamma_{2,n_2+1})$, we obtain from the proposition 1.5 other parametrizations φ_1 and φ_2 for I_1 and I_2, respectively, namely

$$A_1 = \left(\begin{array}{c|c} A_1' & \mathbf{0} \\ \hline \alpha_1 & \gamma \end{array}\right) \text{ et} A_2 = \left(\begin{array}{c|c} A_2' & \mathbf{0} \\ \hline \alpha_2 & \gamma \end{array}\right),$$

repectively, where $\alpha_1 = \frac{\gamma}{\gamma_{1,n_1+1}} \cdot \alpha_1' \in \mathbb{Z}^{n_1}$ et $\alpha_1 = \frac{\gamma}{\gamma_{1,n_2+1}} \cdot \alpha_2' \in \mathbb{Z}^{n_2}$. Let $\overline{\varphi} : \Bbbk[\mathbf{z}, x] \to \Bbbk[\mathbf{t}^{\pm 1}, \mathbf{u}^{\pm 1}, s^{\pm 1}]$ be the morphism defined by $\overline{\varphi}(x) = s^{\gamma}$ and $\overline{\varphi}(z_{j,i}) = \varphi_i(z_{j,i})$ où $i = 1, 2$.

Now we are able to prove the theorem:

1. We set $J := \text{Ker } \overline{\varphi}$, then J is a toric ideal and we have that:
   - J is homogeneous. This is obtained by the following reasoning:
     Since I_1 and I_2 are homegeneous ideal, by the lemma 2.2, there exist $\omega_1 \in \mathbb{Q}^{m_1}$, $\omega_2 \in \mathbb{Q}^{m_2}$, $\omega' \in \mathbb{Q}$ and $\omega' \in \mathbb{Q}$ such that

     $$\begin{aligned} (1, \ldots, 1) &= (\omega_1, \omega') \cdot A_1 = (\omega_1 \cdot A_1' + \omega' \cdot \alpha_1, \omega' \cdot \gamma) \text{ et} \\ (1, \ldots, 1) &= (\omega_2, \omega'') \cdot A_2 = (\omega_2 \cdot A_2' + \omega'' \cdot \alpha_2, \omega'' \cdot \gamma). \end{aligned}$$

     Thus $\omega' \cdot \gamma = \omega'' \cdot \gamma$ and by consequence $\omega' = \omega''$ Moreover, the matrix which represents the morphism $\overline{\varphi}$ is

     $$A = \left(\begin{array}{c|c|c} A_1' & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & A_2' & \mathbf{0} \\ \hline \alpha_1 & \alpha_2 & \gamma \end{array}\right)$$

     and

     $$\begin{aligned} (\omega_1, \omega_2, \omega') \cdot A &= (\omega_1 \cdot A_1 + \omega' \cdot \alpha_1, \omega_2 \cdot A_2 + \omega' \cdot \alpha_2, \omega'\gamma) \\ &= (\omega_1 \cdot A_1 + \omega' \cdot \alpha_1, \omega_2 \cdot A_2 + \omega' \cdot \alpha_2, \omega'\gamma) \\ &= (1, \ldots, 1, 1, \ldots, 1, 1). \end{aligned}$$

     Due to lemma 2.2, $J = \text{Ker } \overline{\varphi} = I_A$ is homogeneous.
   - Let us prove that $J = I_1 + I_2$. It is clear that $I_1 + I_2 \subset J$. We need to prove that $J \subset I_1 + I_2$: Let $B := \mathbf{z_1}^{u_1} \mathbf{z_2}^{u_2} x^{\delta} - \mathbf{z_1}^{v_1} \mathbf{z_2}^{v_2} \in J$ homogeneous, where $\delta \in \mathbb{N}$ and $u_i, v_i \in \mathbb{N}^{n_i}$ for any $i = 1, 2$, with the property that if $(u_i)_j \neq 0$, then $(v_i)_j = 0$ and reciprocally. Thus

     $$|u_1| + |u_2| + \delta = \deg_{\mathbb{N}}(\mathbf{z_1}^{u_1} \mathbf{z_2}^{u_2} x^{\delta}) = \deg_{\mathbb{N}}(\mathbf{z_1}^{v_1} \mathbf{z_2}^{v_2}) = |v_1| + |v_2|. \quad (2)$$

     Furthermore, since

     $$0 = \overline{\varphi}(B) \widetilde{\varphi_1}(\mathbf{z_1}^{u_1}) \widetilde{\varphi_2}(\mathbf{z_2}^{u_2}) s^{\lambda_1} - \widetilde{\varphi_1}(\mathbf{z_1}^{v_1}) \widetilde{\varphi_2}(\mathbf{z_2}^{v_2}) s^{\lambda_2}$$

     and the variable sets $\mathbf{t}$, $\mathbf{w}$ and s are disjoint,

     $$\widetilde{\varphi_1}(\mathbf{z_1}^{u_1}) = \widetilde{\varphi_1}(\mathbf{z_1}^{v_1}) \text{ et } \widetilde{\varphi_2}(\mathbf{z_2}^{u_2}) = \widetilde{\varphi_2}(\mathbf{z_2}^{v_2}) \text{ et } \lambda_1 = \lambda_2.$$

     So $\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1} \in (\text{Ker } \widetilde{\varphi_1})$ et $\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2} \in (\text{Ker } \widetilde{\varphi_2})$. By the lemma 2.3:

     $$(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} \in (\text{Ker } \widetilde{\varphi_1})^{\text{hom}} = I_1 \text{ et } (\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} \in (\text{Ker } \widetilde{\varphi_2})^{\text{hom}} = I_2.$$

     - We will prove that B is a algebric combination of $(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}}$ and $(\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}}$. In order to prove it, we consider three cases:
       (a) Let $|u_1| < |v_1|$. Then, there exists $\alpha \in \mathbb{N}$, $\alpha > 0$, such that

       $$(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} = \mathbf{z_1}^{u_1} x^{\alpha} - \mathbf{z_1}^{v_1}.$$

       So

       $$|u_1| + \alpha = |v_1|. \quad (3)$$

       Now, we consider the following two subcases:

i. Let $|u_2| \leq |v_2|$. Then, there exist $\beta \in \mathbb{N}$, such that

$$(\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} = \mathbf{z_2}^{u_2} x^{\beta} - \mathbf{z_2}^{v_2}.$$

Then $|u_2| + \beta = |v_2|$. Furthermore, from the equations 3 and 2, we deduce that

$$(|u_1| + \alpha) + (|u_2| + \beta) = |v_1| + |v_2| = |u_1| + |u_2| + \delta.$$

By consequence $\alpha + \beta = \delta$ and

$$\begin{aligned} B &= \mathbf{z_1}^{u_1}\mathbf{z_2}^{u_2}x^{\delta} - \mathbf{z_1}^{v_1}\mathbf{z_2}^{v_2} \\ &= \mathbf{z_1}^{u_1}x^{\alpha}(\mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2}) + \mathbf{z_2}^{v_2}(\mathbf{z_1}^{u_1}x^{\alpha} - \mathbf{z_1}^{v_1}) \\ &= \mathbf{z_1}^{u_1}x^{\alpha}(\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} + \mathbf{z_2}^{v_2}(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} \in I_2 + I_1. \end{aligned}$$

ii. Let $|u_2| > |v_2|$. Then, there exists $\beta \in \mathbb{N}$, such that

$$(\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} = \mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2} x^{\beta}.$$

Thus $|u_2| = |v_2| + \beta$. Furthermore, from the equation 3 and 2, we deduce that

$$(|u_1| + \alpha) + |u_2| = |v_1| + (|v_2| + \beta) = (|v_1| + |v_2|) + \beta = (|u_1| + |u_2| + \delta) + \beta$$

So $\alpha = \delta + \beta$ and

$$\begin{aligned} B &= \mathbf{z_1}^{u_1}\mathbf{z_2}^{u_2}x^{\delta} - \mathbf{z_1}^{v_1}\mathbf{z_2}^{v_2} \\ &= \mathbf{z_1}^{u_1}x^{\delta}(\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2}x^{\beta}) + \mathbf{z_2}^{v_2}(\mathbf{z_1}^{u_1}x^{\alpha} - \mathbf{z_1}^{v_1}) \\ &= \mathbf{z_1}^{u_1}x^{\delta}(\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} + \mathbf{z_2}^{v_2}(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} \in I_2 + I_1. \end{aligned}$$

(b) Let $|u_1| > |v_1|$. Then, from the equation 2 $|v_2| > |u_2|$. By this manner, there exist $\alpha, \beta \in \mathbb{N}$, $\alpha > 0$ and $\beta > 0$ such that

$$(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} = \mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1}x^{\alpha} \text{ et } (\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} = \mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2}.$$

So $|u_1| + (|u_2| + \beta) = (|v_1| + \alpha) + |v_2| = (|v_1| + |v_2|) + \alpha = (|u_1| + |u_2| + \delta) + \alpha$, this last equality is due to the equation 2. By consequence $\beta = \delta + \alpha$ and finally we have:

$$\begin{aligned} B &= \mathbf{z_1}^{u_1}\mathbf{z_2}^{u_2}x^{\delta} - \mathbf{z_1}^{v_1}\mathbf{z_2}^{v_2} \\ &= \mathbf{z_2}^{u_2}x^{\delta}(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1}x^{\alpha}) + \mathbf{z_2}^{v_1}(\mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2}) \\ &= \mathbf{z_2}^{u_2}x^{\delta}(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} + \mathbf{z_2}^{v_1}(\mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2})^{\text{hom}} \in I_1 + I_2. \end{aligned}$$

(c) Let $|u_1| = |v_1|$. Then, from the equation 2, $|v_2| \geq |u_2|$. By this way, there exist $\beta \in \mathbb{N}$, such that

$$(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} = \mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1} \text{ et } (\mathbf{z_2}^{u_2} - \mathbf{z_2}^{v_2})^{\text{hom}} = \mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2}.$$

Thus $|u_1| + (|u_2| + \beta) = |v_1| + |v_2| = |v_1| + |v_2| = |u_1| + |u_2| + \delta$, this last equality is due to the equation 2. By consequence $\beta = \delta$ and we have:

$$\begin{aligned} B &= \mathbf{z_1}^{u_1}\mathbf{z_2}^{u_2}x^{\delta} - \mathbf{z_1}^{v_1}\mathbf{z_2}^{v_2} \\ &= \mathbf{z_2}^{u_2}x^{\delta}(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1}) + \mathbf{z_2}^{v_1}(\mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2}) \\ &= \mathbf{z_2}^{u_2}x^{\delta}(\mathbf{z_1}^{u_1} - \mathbf{z_1}^{v_1})^{\text{hom}} + \mathbf{z_2}^{v_1}(\mathbf{z_2}^{u_2}x^{\beta} - \mathbf{z_2}^{v_2})^{\text{hom}} \in I_1 + I_2. \end{aligned}$$

It proves that $J \subset I_1 + I_2$ and we conclude $I_1 + I_2 = J := \text{Ker } \overline{\varphi}$.

Finally, by the remark 1.1 we conclude $I_1 + I_2$ is a toric homogeneous ideal.

2. We have proved that the matrix $A \in M_{m_1+m_2+1,n_1+n_2+1}(\mathbb{Z})$ which represents the parametrization $\overline{\varphi}$ of $I_1 + I_2$ is given by:
$$A = \left(\begin{array}{c|c|c} A_1' & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & A_2' & \mathbf{0} \\ \hline \alpha_1 & \alpha_2 & \gamma \end{array}\right).$$
By consequence: $\dim\ (\Bbbk[\mathbf{z_1}, \mathbf{z_2}, x]/(I_1 + I_2)) = \dim\ (\Bbbk[\mathbf{z_1}, x]/I_1) + \dim\ (\Bbbk[\mathbf{z_2}, x]/I_2) - 1$.

■

2.7 Proposition. *Let $\mathbf{z_1}, \ldots, \mathbf{z_k}$ be k disjoint sets of variables and for any $i = 1, \ldots, k$, let I_i be a homogeneous toric ideal with the maximal rank parametrizations $\varphi_i : \Bbbk[\mathbf{z_i}] \to \Bbbk[\mathbf{t_i}^{\pm 1}]$. Then $\sum_{i=1}^{k} I_i$ is a homogeneous toric ideal with maximal rank parametrization*

$$\varphi : \Bbbk[\mathbf{z_1}, \ldots, \mathbf{z_k}] \to \Bbbk[\mathbf{t_1}^{\pm 1}, \ldots, \mathbf{t_k}^{\pm 1}] \text{ the morphism defined by } \varphi(z_{i,j}) = \varphi_i(z_{i,j}).$$

Moreover

$$\dim\ (k[\mathbf{z_1}, \ldots, \mathbf{z_k}]/(\sum_{i=1}^{k} I_i)) = \sum_{i=1}^{k} \dim\ (\Bbbk[\mathbf{z_i}]/I_i).$$

Proof. We set $J := \sum_{i=1}^{k} I_i$. It is clear that $J \subset \mathrm{Ker}\ \varphi$. We will prove that $\mathrm{Ker}\ \varphi \subset J$:
Let m_1, m_2 be monomials of $\Bbbk[\mathbf{z_1}, \ldots, \mathbf{z_k}]$ such that $b = m_1 - m_2$ is a generator of $\mathrm{Ker}\ \varphi$. Furthermore, we write $m_1 = m_1^{(1)} m_1^{(2)} \cdots m_1^{(k)}$ and $m_2 = m_2^{(1)} m_2^{(2)} \cdots m_2^{(k)}$, where for any $1 \le i \le k$ $m_1^{(i)}$ and $m_2^{(i)}$ are monomials in $\Bbbk[\mathbf{z_i}]$. Since $\varphi(m_1) = \varphi(m_2)$ and $\mathbf{t_1}, \ldots, \mathbf{t_k}$ are disjoint sets of variable, $\varphi(m_1^{(i)}) = \varphi(m_2^{(i)})$. So, for any $1 \le i < j \le k$:

$$\varphi_i(m_1^{(i)}) = \varphi(m_1^{(i)}) = \varphi(m_2^{(i)}) = \varphi_i(m_2^{(i)}),$$

thus $m_1^{(i)} - m_2^{(i)} \in \mathrm{Ker}\ \varphi_i = \mathcal{I}_i$ and

$$m_1 - m_2 = m_1^{(2)} \cdots m_1^{(k)} (m_1^{(1)} - m_2^{(1)}) + m_2^{(1)} m_1^{(3)} \cdots m_1^{(k)} (m_1^{(2)} - m_2^{(2)}) + \cdots + m_2^{(1)} \cdots m_2^{(k-1)} (m_1^{(k)} - m_2^{(k)}) \in J.$$

Thus $\mathrm{Ker}\ \varphi \subset J$ and we conclude that $J = \mathrm{Ker}\ \varphi$. So J is a toric ideal. It is clear that the parametrizatrion φ of J is a maximal rang parametrization, thus

$$\dim\ (k[\mathbf{z_1}, \ldots, \mathbf{z_k}]/(\sum_{i=1}^{k} I_i)) = \sum_{i=1}^{k} \dim\ (\Bbbk[\mathbf{z_i}]/I_i).$$

■

2.8 Definition. Let $\mathbf{x_1}, \ldots, \mathbf{x_k}$ be variable sets and $I_1 \subset \Bbbk[\mathbf{x_1}], \ldots, I_k \subset \Bbbk[\mathbf{x_k}]$ be toric ideals such that for all pair $i, j \in \{1, \ldots, k\}$, $i \neq j$, $|\mathbf{x_i} \cap \mathbf{x_j}| \le 1$. We define the **graph of the sequence of toric ideals** $I_1, \ldots, I_k$, denoted by $G(I_1, \ldots, I_k)$,as the graph whose vertice set is $I_1, \ldots, I_k$ and $\{I_i, I_j\}$ is an edge if I_i and I_j has a common variable.

2.9 Theorem. *Let $\mathbf{x_1}, \ldots, \mathbf{x_k}$ be set of variables and $I_1 \subset \Bbbk[\mathbf{x_1}], \ldots, I_k \subset \Bbbk[\mathbf{x_k}]$ be toric ideals such that for any pair $i, j \in \{1, \ldots, k\}$, $i \neq j$, $|\mathbf{x_i} \cap \mathbf{x_j}| \le 1$. If for any connected component $G(I_1, \ldots, I_k)$ is a tree, then $J := \sum_{i=1}^{k} I_i$ is a homogeneous toric ideal and setting r as the number of connected components of $G(I_1, \ldots, I_k)$ we have that:*

$$\dim\ (\Bbbk[\mathbf{x}]/J) = \sum_{i=1}^{k} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) + r - k + 1,$$

where $\mathbf{x} = \cup_{i=1}^{k} \mathbf{x_i}$.

Proof. Let G_j be a connected component of $G := G(I_1, \dots, I_k)$ and $V(G_j) = \{I_{j_1}, \dots, I_{j_{s_j}}\}$ be the vertex set of G_j. We remark that $J := \sum_{i=1}^{k} I_i = \sum_{j=1}^{r} (\sum_{i=1}^{s_j} I_{j_i})$. So, if we prove that for any $j \in \{1, \dots, r\}$, $(\sum_{i=1}^{s_j} I_{j_i})$ is a homogeneous toric ideal and

$$\dim\ (\Bbbk[\mathbf{x}]/\sum_{i=1}^{s_j} I_{j_i}) = \sum_{i=1}^{s_j} \dim\ (\Bbbk[\mathbf{x_{j_i}}]/I_i) - s_j,$$

then J is a homogeneous toric ideal due to the proposition 2.7 and

$$\dim\ (\Bbbk[\mathbf{x}]/J) = \sum_{i=1}^{k} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) + r - k + 1.$$

Then, we can assume that G is a connected tree and we will prove by induction over k that J is a homogenous toric ideal.

1. If $k = 2$, it is the theorem 2.6
2. We assume that if G is a connected tree with $k-1$-vertices, then $\sum_{i=1}^{k-1} I_i$ is a toric homogeneous ideal and
$$\dim\ (\Bbbk[\mathbf{x}]/\sum_{i=1}^{k-1} I_i) = \sum_{i=1}^{k-1} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) - (k-1).$$
We must prove that if G is a connected tree with k-vertice, then $J := \sum_{i=1}^{k} I_i$ is a toric homogeneous ideal and
$$\dim\ (\Bbbk[\mathbf{x}]/J) = \sum_{i=1}^{k} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) - k.$$
It will follow from the next argument: since G is a connected tree, there exists $i \in \{1, \dots, k\}$ such that there exists a unique $j \in \{1, \dots, k\}$ such that $\{I_i, I_j\} \in E(G)$. We can assume that $i = k$, thus $|\mathbf{x_i} \cap \mathbf{x_k}| = 1$ and for any $j \in (\{1, \dots, k-1\} \setminus \{i\})$, $|\mathbf{x_j} \cap \mathbf{x_k}| = 0$. So $|(\cup_{i=1}^{k-1} \mathbf{x_i}) \cap \mathbf{x_k}| = 1$ and $G \setminus \{I_k\}$ is a connected tree. By induction hypothesis $\sum_{i=1}^{k-1} I_i$ is a homogeneous toric ideal and
$$\dim\ (\Bbbk[\mathbf{x}]/\sum_{i=1}^{k-1} I_i) = \sum_{i=1}^{k-1} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) - (k-1).$$
In addition, I_k is a homogeneous toric ideal, so by the theorem 2.6 $J = \sum_{i=1}^{k-1} I_i + I_k$ is a homogeneous toric ideal and
$$\begin{aligned} \dim\ (\Bbbk[\mathbf{x}]/J) &= \dim\ (\Bbbk[\mathbf{x}]/\textstyle\sum_{i=1}^{k-1} I_i) + \dim\ (\Bbbk[\mathbf{x_k}]/I_k) \\ &= \textstyle\sum_{i=1}^{k-1} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) - (k-1) + \dim\ (\Bbbk[\mathbf{x_k}]/I_k) - 1 \\ &= \textstyle\sum_{i=1}^{k} \dim\ (\Bbbk[\mathbf{x_i}]/I_i) - k. \end{aligned}$$

■

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